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by

Aihua W. Shaker

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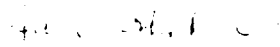
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# On Singular Semilinear Elliptic Equations

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**Abstract:** - For the semilinear elliptic equation  $\Delta u + p(x)u^{-\gamma} = 0$ ,  $x \in R^n$ ,  $n \geq 3$ ,  $\gamma > 0$ , we show via the barrier method the existence of a positive entire solution behaving like  $|x|^{2-n}$  near  $\infty$ .

## 1 Introduction

We study the singular semilinear elliptic equation

$$(1) \quad \Delta u + p(x)u^{-\gamma} = 0$$

in  $R^n$ . This type of equation arises in the boundary layer theory of viscous fluids [3,4]. From the results of Fulks and Maybee [8], Crandall, Rabinowitz, and Tartar [5], Gomes [9], and recently Lazer and McKenna [14], it follows that (1) has a unique classical solution within a bounded domain  $\Omega$ , where  $p(x)$  is a sufficiently regular function which is positive on  $\Omega$ . Kusano and Swanson [12] gave the existence proof on exterior domains. As for the existence of entire solutions, not much is known. Edelson [7], Kusano and Swanson [13] have been able to show the existence of entire solutions of (1) with  $\gamma \in (0, 1)$ , and  $p(x)$  sufficiently regular. In this paper we show via the upper and lower solution method, which is also referred to as the barrier method, that (1) has a bounded positive entire solution vanishing at  $\infty$  in  $R^n$  for  $n \geq 3$  and all  $\gamma > 0$ .

The author learned after this paper was finished that a similar result was given earlier by R. Dalmasso [6], but by a different approach.

## 2 Preliminaries

We first state the theorem by Kusano and Swanson [13] for the case  $0 < \gamma < 1$ .

**Lemma 1.** *Equation (1) has an entire bounded positive solution  $u(x)$  in  $R^n$  for  $n \geq 3$ , and  $|x|^{n-2}u(x)$  is bounded and bounded away from zero near  $\infty$  if  $p(x)$  satisfies the following conditions:*

(H1)  $p(x) \in C_{loc}^\alpha(R^n)$ ,  $n \geq 3$ ,  $p(x) > 0$ ,  $x \in R^n \setminus \{0\}$ ,

(H2)  $\exists C > 0$ , such that  $C\phi(|x|) \leq p(x) \leq \phi(|x|)$ ,  $\phi(x) = \max_{|x|=t} p(x)$ ,  $0 \leq t \leq \infty$ ,

(H3)  $\int_1^\infty t^{n-1+\gamma(n-2)}\phi(t)dt < \infty$ .

The term "entire" has often been used for solutions of equation (1) in  $R^n$ . To avoid confusion with the traditional definition for entire functions, we use the term " $C^{2+\alpha}$ -entire". A  $C^{2+\alpha}$ -entire solution of (1) is defined to be a function  $u(x) \in C_{loc}^{2+\alpha}(R^n)$  that satisfies (1) pointwise in  $R^n$ .

The method that we shall be using heavily in our proof is the so-called **barrier method**, or **upper-lower solution method**.

We consider the elliptic boundary value problem

$$(2) \quad \begin{cases} Lu + f(x, u) = 0 & \text{in } D \\ Bu = a\partial u / \partial \nu + bu = g & \text{on } \partial D \end{cases}$$

where  $D$  is a smoothly bounded domain in  $R^N$  and  $\nu = (\nu_1, \dots, \nu_n)$  is a smoothly varying outward normal vector field on  $\partial D$  which is of class  $C^{2+\alpha}$ , while  $a$  and  $b$  are positive constants. We also assume that  $f \in C^\alpha$  and that  $g$  has an extension  $\hat{g}$  to the interior of  $D$  such that  $\hat{g} \in C^{2+\alpha}$ .

An **upper solution** to the above problem is a function  $\phi$  satisfying

$$\begin{cases} L\phi + f(x, \phi) \leq 0 & \text{in } D \\ B\phi \geq g & \text{on } \partial D. \end{cases}$$

A **lower solution** to the above problem is a function  $\psi$  satisfying

$$\begin{cases} L\psi + f(x, \psi) \geq 0 & \text{in } D \\ B\psi \leq g & \text{on } \partial D. \end{cases}$$

We assume that  $\partial D, f, g$ , and the coefficients of  $L$  are smooth in what follows

**Lemma 2.** (Theorem 2.3.1 of [16]) Let  $\phi$  be an upper solution and  $\psi$  a lower solution with  $\psi \leq \phi$  on  $D$ . Then there exists a solution  $u$  to the above boundary value problem with  $\psi \leq u \leq \phi$ .

We consider the following example:

$$\begin{cases} u'' + \lambda u - u^3 = 0 & x \in (0, \pi) \\ u = 0 & x = 0, \pi \end{cases}$$

By the above theorem, if  $\lambda > 1$ , then the problem has at least three solutions.

Actually,  $\underline{u} = \epsilon \sin x$  with  $\epsilon$  small is a lower solution, and  $\bar{u} = Rx^{1/2}$  with  $R$  large is an upper solution. Therefore there exists a solution  $u$  such that  $\underline{u} \leq u \leq \bar{u}$  in  $(0, \pi)$ . Clearly  $-u$  and  $0$  are also solutions to this problem.

The following lemma on the barrier method for  $D = R^n$  is due to Ni [15] in 1982. A special case was proved earlier by Ako and Kusano [1] in 1964. The proof is standard. Using the well known result on the upper-lower solution approach in bounded regions (see Sattinger [16]), we first solve the equation

$$Lu + F(x, u) = 0$$

on  $B_R$ . Then by letting,  $R \rightarrow \infty$ , we obtain a solution on  $R^n$  by a diagonal process.

**Lemma 3.** Let  $u_1 \geq u_2$  in  $R^n$  be such that

$$(3) \quad \begin{cases} Lu_1 + f(x, u_1) \leq 0 \\ Lu_2 + f(x, u_2) \geq 0 \end{cases}$$

where  $f$  is locally Hölder continuous in  $(x, u)$  and locally Lipschitz in  $u$ , and  $L$  is an elliptic operator of second order. Then there exists a solution  $u$  of  $Lu + f(x, u) = 0$  with  $u_1 \geq u \geq u_2$ .

### 3 Main Result

**Theorem 1.** Under the same conditions as given in Lemma 1, the equation (1) has a  $C^{2+\alpha}$ -entire positive solution in  $R^N$ ,  $N \geq 3$ , vanishing at  $\infty$  at the rate of at least  $|x|^{q(N-2)}$  with some  $q \in (0, 1)$  for any  $\gamma > 0$ .

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The difficulty in constructing the proof is to find an appropriate upper solution to equation (1). In order to use the barrier method we first study the nonsingular equation

$$\Delta u + p(x)[\delta + u]^{-\gamma} = 0.$$

For each fixed  $\gamma$  there corresponds a solution  $u_\gamma(x)$ . Letting  $\gamma \rightarrow \infty$ , we show that the limiting function is the desired solution.

**Proof:** By Lemma 1, for  $\gamma = \gamma_1 \in (0, 1)$ , equation (1) has a  $C^{2+\alpha}$ -entire positive solution  $u_1(x)$  in  $R^n$ ,  $n \geq 3$ , vanishing at  $\infty$  at the rate  $r^{2-n}$ . We claim that  $\bar{u} = cu_1^q$  is an upper solution of the equation (1) for  $\gamma \geq 1$ , where

$$q < \frac{1 + \gamma_1}{1 + \gamma} < 1,$$

$$c > \left( \frac{M^{1+\gamma_1-q(1+\gamma)}}{q} \right)^{\frac{1}{1+\gamma}}, \quad M = \max_{x \in R^n} |u(x)|.$$

In fact:

$$\begin{aligned} & \Delta \bar{u} + \frac{p(x)}{\bar{u}^\gamma} \\ &= cq(q-1)u^{q-2}|\nabla u|^2 - cq u^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\ &\leq -cq u^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\ &= \frac{p(x)}{c^\gamma u^{\gamma q}} \left( 1 - \frac{c^{1+\gamma} q}{u^{1+\gamma_1-q(1+\gamma)}} \right) \\ &\leq \frac{p(x)}{c^\gamma u^{\gamma q}} \left( 1 - \left( \frac{M^{1+\gamma_1-q(1+\gamma)}}{q} \right)^{\frac{q}{1+\gamma}} \frac{q}{u^{1+\gamma_1-q(1+\gamma)}} \right) \\ &\leq \frac{p(x)}{c^\gamma u^{\gamma q}} (1 - 1) = 0. \end{aligned}$$

Let  $\delta$  be a fixed positive number. We then observe that  $\bar{u}$  is an upper solution of the equation

$$(4) \quad \Delta u(x) + p(x)[u(x) + \delta]^{-\gamma} = 0, \quad x \in R^n.$$

$\underline{u} = 0$  is a lower solution of (4). Since  $\bar{u} = cu_1^q > 0$ ,  $\bar{u} \geq \underline{u}$  in  $R^n$ . By Lemma 2, (4) has a solution  $u$  such that  $\underline{u} \leq u \leq \bar{u}$ .

For  $\hat{\delta} < \delta$ ,  $u$  is a lower solution of (4) with  $\delta = \hat{\delta}$ . Lemma 2 then implies that (4) has a solution  $\hat{u}$  for  $\delta = \hat{\delta}$  such that  $\underline{u} \leq \hat{u} \leq \bar{u}$ .

Let  $\{\delta_n\}_1^\infty$  be a sequence of strictly decreasing positive numbers, and let  $u_n(x)$  be a smooth positive solution of (4) when  $\delta = \delta_n$ . From the construction of our lower solutions, it is clear that  $u_n(x) \geq u_{n-1}(x)$  for all  $n$ . So  $\lim_{n \rightarrow \infty} u_n(x) = u(x)$  exists for all  $x \in R^n$  and

$$(5) \quad \underline{u} \leq u \leq \bar{u}$$

for  $x \in R^n$ .

We can now assert that  $u \in C^{2+\alpha}(R^n)$  and that

$$(6) \quad \Delta u + p(x)u^{-\gamma} = 0$$

for  $x \in R^n$ . This follows from more or less standard arguments.

Let  $x_o \in R^n$  and  $r > 0$ . We consider the ball of radius  $r$  centered at  $x_o$ ,  $B(x_o, r)$  in  $R^n$ . Let  $\Psi$  be a  $C^\infty$  function which is equal to 1 on  $\overline{B(x_o, r/2)}$  and equal to 0 off  $B(x_o, r)$ . We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n$$

for  $n \geq 1$ , where  $p_n$  is a term whose  $L^\infty$  norm is bounded independently of  $n$ . Therefore for  $n \geq 1$  we have

$$\Psi u_n \Delta(\Psi u_n) = \sum_{j=1}^N b_{nj} \frac{\partial(\Psi u_n)}{\partial x_j} + q_n,$$

where  $b_{nj}, j = 1, \dots, n$  and  $q_n$  are terms bounded independently of  $n$  for  $n \geq 1$ . Integrating the above equation, we have that there exist constants  $c_1 > 0$  and  $c_2 > 0$  independent of  $n$  such that

$$\int_{B(x_o, r)} |\nabla u_n|^2 dx \leq c_1 \left( \int_{B(x_o, r)} |\nabla u_n|^2 dx \right)^{1/2} + c_2.$$

From this, it follows that the  $L^2(B(x_o, r))$ -norm of  $|\nabla \Psi u_n|$  is bounded independently of  $n$ . Hence, the  $L^2(B(x_o, r/2))$ -norm of  $|\nabla u_n|$  is bounded independently of  $n$ . Let  $\Psi_1$  be a  $C^\infty$  function which is equal to 1 on  $\overline{B(x_o, r/4)}$  and equal to 0 off  $B(x_o, r/2)$ . We have for  $n \geq 1$ ,

$$\Delta(\Psi_1 u_n) = 2\nabla\Psi_1 \cdot \nabla u_n + p_{1n}.$$



where  $p_{1n}$  is a term whose  $L^\infty(B(x_o, r/2))$ -norm is bounded independently of  $n$ . From standard elliptic theory, the  $W^{2,2}(B(x_o, r/2))$ -norm of  $\Psi_1 u_n$  is also bounded independently of  $n$  and hence, the  $W^{2,2}(B(x_o, r/4))$ -norm of  $u_n$  is bounded independently of  $n$ . Since the  $W^{1,2}(B(x_o, r/4))$ -norm of the components of  $\nabla u_n$  are bounded independently of  $n$ , it follows from the Sobolev embedding theorem that if  $q = 2n/(n-2) > 2$  for  $n > 2$  and in addition if  $q > 2$  is arbitrary for  $n \leq 2$ , then the  $L^q(B(x_o, r/4))$ -norm of  $|u_n|$  is bounded independently of  $n$ . Let  $\Psi_2$  be a  $C^\infty$  function which is equal to 1 on  $\overline{B(x_o, r/8)}$  and equal to 0 off  $B(x_o, r/4)$ . We have for  $n \geq 1$ ,

$$\Delta(\Psi_2 u_n) = 2\nabla \Psi_2 \cdot \nabla u_n + p_{2n},$$

where  $p_{2n}$  is a term whose  $L^\infty(B(x_o, r/4))$ -norm is bounded independently of  $n$ . Since the right hand side of the above equation is bounded in  $L^q(B(x_o, r/4))$  independently of  $n$ , the  $W^{2,q}(B(x_o, r/4))$ -norm of  $\Psi_2 u_n$  is also bounded independently of  $n$ . Hence, the  $W^{2,q}(B(x_o, r/8))$ -norm of  $u_n$  is bounded independently of  $n$ . Continuing this line of reasoning, after a finite number of steps, we find a number  $r_1 > 0$  and  $q_1 > n/(1-\alpha)$  such that the  $W^{2,q_1}(B(x_o, r_1))$ -norm of  $u_n$  is bounded independently of  $n$ . Hence, there is a subsequence of  $\{u_n\}_1^\infty$ , which we may assume is the sequence itself, which converges in  $C^{1+\alpha}(\overline{B(x_o, r_1)})$ . If  $\theta$  is a  $C^\infty$  function which is equal to 1 on  $\overline{B(x_o, r_1/2)}$  and 0 off  $B(x_o, r_1)$ , then

$$\Delta(\theta u_n) = 2\nabla \theta \cdot \nabla u_n + \hat{p}_n, \text{ where } \hat{p}_n = \theta \Delta u_n + u_n \Delta \theta.$$

The right-hand side of the above equation converges in  $C^\alpha(\overline{B(x_o, r_1)})$ . Hence by Schauder theory,  $\{\theta u_n\}_1^\infty$  converges in  $C^{2+\alpha}(\overline{B(x_o, r_1)})$  and thus  $\{u_n\}_1^\infty$  converges in  $C^{2+\alpha}(\overline{B(x_o, r_1/2)})$ . Since  $x_o$  was arbitrary, this shows that  $u \in C^{2+\alpha}(R^n)$ . Clearly (6) holds.

#### 4 Some remarks

##### Remark 1:

For  $n = 1$ , the properties of positive solutions of equation (1) have been studied by Taliaferro [17], and Gatica [10]. For  $n = 2$ , no entire positive solution of equation (1) exists regardless of its asymptotic behavior at  $\infty$  (see [13]).

##### Remark 2:

It is observed by Callegari, Friedman and Nachman [2], [3,4] that if the partial differential equations describing the boundary layer behind a rarefaction or shock wave (with viscosity proportional to the temperature) traveling down, and perpendicular to, a flat plate are written in terms of a stream function and a similarity variable the following Blasius-type equation emerges [18].

$$f'''(\eta) + f(\eta)f''(\eta) = 0,$$

where

$$f(0) = 0, \quad f'(0) = K, \quad f'(\infty) = 1.$$

Here,  $0 < K < 1$ , for rarefaction waves and,  $1 < K < 6$ , for shock waves. ( $K = 0$  corresponds to the classical Blasius problem.) Adopting the Crocco variables

$$x = f'(\eta), \quad g = f''(\eta)$$

results in the system

$$gg'' + x = 0,$$

$$g'(K) = 0, \quad g(1) = 0,$$

which falls into the class of equation discussed in this paper.

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